Spectral properties of small dusty clusters

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A monolayer dusty crystal can be observed in the glow discharge. In particular, a small number of dusty grains form simple atomlike plane clusters. Stability and oscillations of the polygonal cluster are considered. For the simplest stable clusters normal displacements and frequencies are found for an arbitrary form of mutual interaction. The measurement of these modes can help to give in detail the form of the potential function.

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I. INTRODUCTION

Numerous industrial applications have triggered active research on the phenomena associated with dust dynamics in low-pressure glow discharge [1]. Dusty plasma is formed by introducing micron-sized grains into plasma. Typically the grains are negatively charged due to higher mobility of electrons as compared to that of ions. Then the dusty grains can be electrically suspended in the sheath above the electrodes, where the gravity is exactly balanced by the electric force.

The suspended dusty particles can be strongly coupled due to their high-negative charge (about 10^4 electrons per micron-sized grain) to form liquid states and ordered lattice structures known as Coulomb crystals [2–7].

The reader should keep in mind that the plasma edge near the electrodes is characterized by highly nonequilibrium conditions. The grains are exposed to a vertical ion flow resulting both in an ion-drag force [8] and specific attractive forces induced by the ions focusing [9]. Then the mutual interaction in a bulk dusty crystal is characterized by a complicated strongly anisotropic potential.

Alternatively, a monolayer dusty crystal can also be induced in a glow discharge. In this case, the grains lie in a two-dimensional (2D) plane normal to the vertical ion flow. Their mutual interaction is much simpler and can be characterized by the repulsive Yukawa potential

$$U(r)=\frac{q^2}{r}\exp\left(-\frac{r}{r_d}\right), \tag{1}$$

where q is the grain charge and r_d is the Debye length. Equation (1) is believed to be a reasonable first approximation. In more complicated cases, it may be accomplished with the terms describing the non-Debye shielding, the shadow force, and other effects. In particular, we will consider a more general potential function containing the short-range repulsion and the long-range attraction,

$$U(r)=\frac{q^2}{r}\exp\left(-\frac{r}{r_d}\right)-\alpha\frac{q^2}{r}, \tag{2}$$

where the parameter α ranges from 0 to 1.

In an infinite 2D crystal, the triangular lattice is the basic stable structure. The opposite case corresponds to a relatively small number of dusty grains. This type of classical systems consisting of a small number of particles interacting through repulsive interactions was observed in recent experiments [10,11]. It can be classified as the strongly coupled atomlike clusters. The grains undergo some external confining force that can be approximated by the parabolic potential,

$$U_{\mathrm{conf}}=\tfrac{1}{2}m\omega_0^2r^2, \tag{3}$$

where m is the grain mass and ω_0 is seen as the frequency of axial oscillations of a single grain. The product $m\omega_0^2$ describes the confining potential curvature. This curvature can be varied in experiments.

Depending on the total number of particles the confining potential's detailed form and mutual repulsion, a complicated structure with the inner triangular cores surrounded by the outer circular shells can be observed. Certain number of particles have multiple stable equilibrium states. If the number of particles is small (from 1 to 5), the grains form an extremely simple polygonal configuration. In this paper we study eigenmodes and spectral properties of such an ''atom.''

The equilibrium dusty crystals of the smallest size are predictable. The grain position can be simulated by a simple analytical model [12]. The dusty polygon perturbation in the direction perpendicular to the polygon's plane has previously been considered [13]. Here attention focuses on the oscillations located in the polygon plane. The goal is to present the results applicable to the arbitrary potential function $U(r)$, provided that a polygonal solution exists. Then the measurements of the eigenmodes and frequencies will allow one to restore appropriate corrections to Eq. (1).

II. BASIC EQUATIONS

Let N point dusty grains with mass m and charge q be exposed to the parabolic potential (3). The equation of motion for the particle n includes the trapping potential contribution and the forces of interaction with the kth particle. We assume that the grains interaction can be described by the potential function $U(r)$. In order to simplify the expression for the mutual forces, we introduce a certain function $f(\xi)$, so to put the potential in the form

$$U(r)\equiv f(r^2).$$

Then the classical equation of motion is

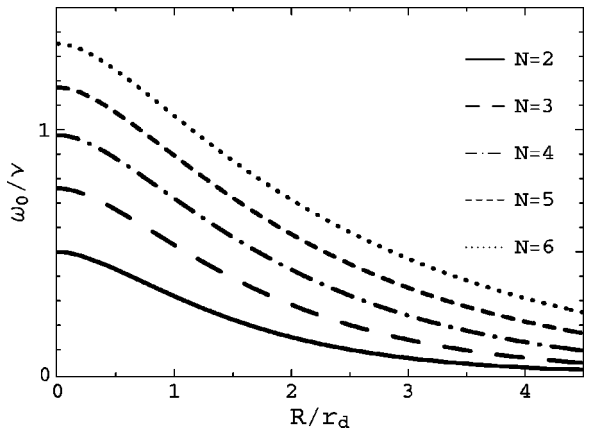

FIG. 1. Trap parameter ω_0 is shown as a function of the equilibrium regular N-gon size R. The frequency ω_0 is measured in units of $\nu=\sqrt{q^2/(mR^3)}$. In contrast with ω_0, the ratio ω_0/ν is finite for small values of R. The polygon size is measured in units of Debye length r_d.

$$\frac{d^2\vec{r}_n}{dt^2}+\omega_0^2\vec{r}_n+\frac{2}{m}\sum_{k\neq n} f'(|\vec{r}_n-\vec{r}_k|^2)(\vec{r}_n-\vec{r}_k)=0. \qquad (4)$$

All particles are assumed to move in the xy plane. We find it useful to describe the grain position by a complex variable $z=x+iy$. The complex version of Eq. (4) is

$$\frac{d^2z_n}{dt^2}+\omega_0^2 z_n+\frac{2}{m}\sum_{k\neq n} f'(|z_n-z_k|^2)(z_n-z_k)=0. \qquad (5)$$

Our initial goal is to obtain the solution corresponding to the regular polygon. To do so we introduce the complex Nth root of 1,

$$\eta=\exp\left(\frac{2\pi i}{N}\right).$$

The sequence η^n, $n=1,2,3,\ldots$ forms a regular polygon in the complex z plane. Let R be the radius of a circle circumscribed about the polygonal cluster. Substituting

$$z_n=R\,\eta^n, \quad 1\leqslant n\leqslant N$$

into Eq. (5), one can see that the equilibrium stationary polygon obeys the restriction

$$\omega_0^2=-\frac{4}{m}\sum_{k=1}^{N-1}\sin^2\left(\frac{\pi k}{N}\right)f'(\xi)\Big|_{\xi=4R^2\sin^2(\pi k/N)}. \qquad (6)$$

Equation (6) should be used to obtain an equilibrium size (i.e., R) of the polygon in the given trapping field. Except for the simple case of power law of the mutual force, this has to be done numerically. Another possibility is to obtain a parabolic potential's curvature required to confine a polygon of fixed size R. Then the condition $\omega_0^2>0$ must be met. In Fig. 1 the ω_0 dependence of R is shown for small polygonal clusters and repulsive Yukawa potential. Note, that ω_0 must tend to infinity in the limit that R becomes small. Thus in Fig. 1 the trap frequency ω_0 is measured in units of $\nu=\sqrt{q^2/(mR^3)}$. The ratio ω_0/ν is finite for all values of R.

The right-hand side of Eq. (6) is an example of the specific trigonometric sum that will frequently appear below. To simplify, we introduce the notation

$$A_l(N)=\frac{4}{m}\sum_{k=1}^{N-1}\sin^2\left(\frac{\pi kl}{N}\right)[-f'(\xi)]_{\xi=4R^2\sin^2(\pi k/N)},$$

where $A_{N\pm l}(N)=A_l(N)$ and $A_l(N)=A_{-l}(N)$. Equation (6) takes the form

$$\omega_0^2=A_1(N). \qquad (7)$$

Similarly, considering the polygon perturbation, it is useful to introduce

$$B_l(N)=\frac{4}{m}\sum_{k=1}^{N-1}\sin^2\left(\frac{\pi kl}{N}\right)[\xi f''(\xi)]_{\xi=4R^2\sin^2(\pi k/N)}$$

with the same symmetry properties.

Now let us consider a small perturbation of the equilibrium polygon. To do so we substitute $z+\delta z$ for z in Eq. (5). More precisely, we put

$$z_n+\delta z_n=\eta^n[R+w_n(t)],$$

where $|w_n(t)|\ll R$. A rather cumbersome equation containing $w_n(t)$ as well as its complex conjugate $\overline{w_n(t)}$ is obtained. The solution can be found as the superposition of two conjugate harmonics,

$$w_n(t)=u(t)\,\eta^{ln}+\overline{v(t)}\,\eta^{-ln}, \qquad (8)$$

where integer l is analogous of the azimuthal wave number. Note, that because of identity $\eta^N\equiv 1$ there are only N different complex harmonics. Thus in Eq. (8) one can assume l ranging from 0 to $\frac{1}{2}N$. Making some efforts yields the following system:

$$\frac{d^2u}{dt^2}+(A_1-A_{l+1}+B_{l+1})u+(B_1-B_l)v=0, \qquad (9)$$

$$\frac{d^2v}{dt^2}+(A_1-A_{l-1}+B_{l-1})v+(B_1-B_l)u=0, \qquad (10)$$

where the argument N is dropped in all trigonometric sums.

In a standard way, we can assume $u,v\sim\exp(i\omega t)$ to obtain the dispersion relation

$$\begin{vmatrix} A_1-A_{l+1}+B_{l+1}-\omega^2 & B_1-B_l \\ B_1-B_l & A_1-A_{l-1}+B_{l-1}-\omega^2 \end{vmatrix}=0. \qquad (11)$$

Equation (11) results in two real values of ω^2. For the stability sake, both eigenvalues should be positive. Then the stability criterion is

$$(A_1-A_{l+1}+B_{l+1})(A_1-A_{l-1}+B_{l-1})\geqslant(B_1-B_l)^2. \tag{12}$$

For given N the criterion must be met for all possible values of l. We investigate this condition for some important cases in the next section.

III. APPLICATIONS

Although our main goal is to investigate small dusty clusters, the approach presented here is general. The only requirement is that Eq. (7) should have a solution describing a stationary polygon. Thus the particles interaction should be repulsive, at least at small distances. If it is the case, our dispersion relation can be applied to any potential function $U(r)$. We start with two simple examples.

A. Vortex rings

Stability of vortex rings is a very old problem first studied by Kelvin [14] and Thomson [15] in the context of the vortex model of an atom. It is well known that in two dimensions the behavior of point vortexes in the ideal fluid is equivalent to dynamics of magnetized electron columns [16]. The interaction of two charged columns can be described by the logarithmic potential $U(r)=2q'^2\ln(1/r)$, where q' should be treated as the charge per unit of column length. We note, that for the one-component nonneutral plasma the external magnetic field acts like the two-dimensional parabolic potential wall in the rotating frame of reference. In this case,

$$A_l(N)=B_l(N)=\frac{q'^2}{mR^2}\sum_{k=1}^{N-1}\frac{\sin^2\left(\frac{\pi kl}{N}\right)}{\sin^2\left(\frac{\pi k}{N}\right)}, \tag{13}$$

where m is the mass per unit of column length. Substituting Eq. (13) into Eq. (12) we obtain the known stability criterion:

$$2(N-1)\geqslant\sum_{k=1}^{N-1}\frac{\sin^2\left(\frac{\pi kl}{N}\right)}{\sin^2\left(\frac{\pi k}{N}\right)}. \tag{14}$$

Recall, that for given vortexes number N the criterion should be met for all l, ranging from 0 to $\frac{1}{2}N$. Surprisingly, the sum in the right-hand side of Eq. (14) can be given in an explicit form,

$$\sum_{k=1}^{N-1}\frac{\sin^2\left(\frac{\pi kl}{N}\right)}{\sin^2\left(\frac{\pi k}{N}\right)}=l(N-l). \tag{15}$$

Using Eqs. (14) and (15) we can easily reproduce Tomson's theorem of vortex rings stability [17].

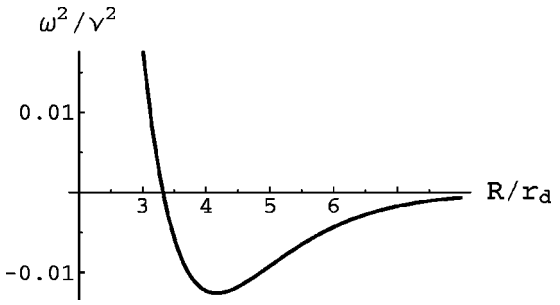

FIG. 2. The squared frequency of the unstable pentagon oscillations against the values of the cluster size R. All variables are normalized as in Fig. 1. The region with $\omega^2<0$ corresponds to instability. The pentagon with $R>3.32r_d$ can be observed only in the event that repulsive Yukawa approximation for grains interaction is incomplete.

B. Trapped ions

Another simple example is a few ions stored in a parabolic trap [18]. The monolayer clusters can be observed in a radio frequency Paul trap [19] under conditions of strong anisotropy. They have been investigated both numerically [20–23] and analytically [23,24]. In this case, $U(r)=q^2/r$ and

$$A_l(N)=\frac{2}{3}B_l(N)=\frac{q^2}{4mR^3}\sum_{k=1}^{N-1}\frac{\sin^2\left(\frac{\pi kl}{N}\right)}{\sin^3\left(\frac{\pi k}{N}\right)}.$$

Substituting these expressions into Eq. (12), we can easily obtain the criterion [24] describing the small ion clusters stability.

C. Dusty grains

Now let us turn to dusty clusters. We assume that mutual repulsion of the grains can be described by Eq. (1). To calculate the trigonometric sums, one should use the function

$$f(\xi)=\frac{q^2}{\sqrt{\xi}}\exp\left(-\frac{\sqrt{\xi}}{r_d}\right). \tag{16}$$

The final form of criterion (12) is rather cumbersome and is not shown. It has been investigated numerically. For $N\leqslant4$, the regular N-gon is stable. The pentagon is stable for $R<3.32r_d$ and unstable otherwise. The unstable mode corresponds to $l=2$. All clusters with $N\geqslant6$ are unstable.

Figure 2 shows the squared frequency of the unstable pentagon oscillations against R. Note that ω^2 goes negative and the system becomes unstable for relatively big clusters. This result can be used to verify the approximation of the repulsive Yukawa potential. The model is incomplete in the event that the regular pentagon with $R>3.32r_d$ can be observed (note that actually the cluster size is twice R). Obviously, in the limit that $R\gg r_d$ the increment is extremely small due to Debye shielding. If it is the case, the instability can be easily overcome by other effects.

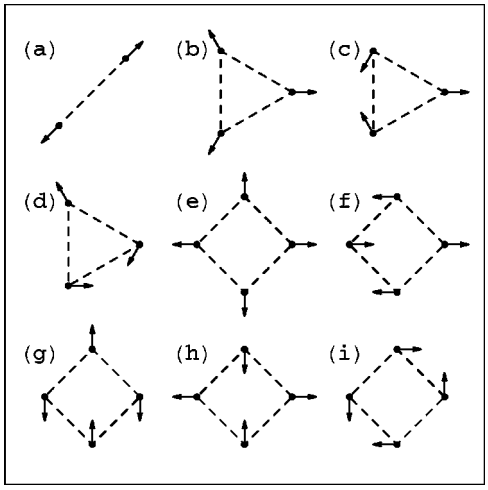

FIG. 3. Sets of normal displacements for equilibrium polygonal clusters. Trivial modes corresponding to a whole cluster shift and rotation are not displayed. For doubly degenerated modes two eigenvectors (c),(d) and (f),(g) correspond to the same frequency. These eigenvectors can be chosen in different ways.

IV. NORMAL DISPLACEMENTS AND FREQUENCIES

In this section, we directly calculate the eigenmodes for the stable polygonal clusters. The frequencies are determined from dispersion relation (11) for an arbitrary form of $f(\xi)$. Then the normal displacements can be found from Eqs. (9) and (10). Alternatively, the normal displacements can *a priori* be found using the representations of the corresponding group of symmetries. We do not give this calculation in detail because it is known from the theory of molecular oscillations [25].

In general, a cluster consisting of N particles must have $3N$ frequencies. We consider a monolayer cluster and focus our attention on the oscillations in the cluster's plane. Then only $2N$ frequencies are left. There always exists a trivial mode with $\omega=0$ corresponding to the rotation of the whole cluster. Another (doubly degenerate) trivial mode corresponds to the shift of the whole cluster. Its frequency $\omega=\omega_0$. Then only $2N-3$ nontrivial modes should be considered. For the simplest clusters, the sets of normal displacements are shown in Fig. 3. Dusty grains are drawn by solid circles. The arrows indicate the direction of the grains motion. Some of the eigenmodes are singly degenerate, such as the radially symmetric polygon oscillation. Others are doubly degenerate and described by two independent sets of normal displacements. Obviously, these two sets can differently be chosen.

A. Two particles

The equilibrium condition becomes

$$\omega_0^2=-\frac{4}{m}f'(4R^2).$$

The only nontrivial mode corresponds to radial oscillations [see Fig. 3(a)]. Its frequency is

$$\omega^2=\frac{32R^2}{m}f''(4R^2).$$

B. Triangle

The equilibrium condition becomes

$$\omega_0^2=-\frac{6}{m}f'(3R^2).$$

The frequency of the radial oscillations [see Fig. 3(b)] is

$$\omega^2=\frac{36R^2}{m}f''(3R^2).$$

In addition, there is a doubly degenerate nontrivial mode [for possible sets of normal displacements see Figs. 3(c) and (d)]. The corresponding frequency is

$$\omega^2=\frac{18R^2}{m}f''(3R^2).$$

C. Square

The equilibrium condition becomes

$$\omega_0^2=-\frac{4}{m}[f'(2R^2)+f'(4R^2)].$$

The frequency of the radial oscillations [see Fig. 3(e)] is

$$\omega^2=\frac{16R^2}{m}[f''(2R^2)+2f''(4R^2)].$$

As for triangle, there is a doubly degenerate mode [for possible sets of normal displacements see Figs. 3(f) and (g)]. The corresponding frequency is

$$\omega^2=\frac{4}{m}[f'(2R^2)-f'(4R^2)+4R^2f''(2R^2)].$$

In addition there are two singly degenerate modes [see Figs. 3(h) and (i)] with

$$\omega^2=\frac{32R^2}{m}f''(4R^2)$$

and

$$\omega^2=\frac{16R^2}{m}f''(2R^2),$$

respectively. In these modes the particles with odd and even numbers move in opposite directions.

The frequencies for the pentagon and the other clusters are rather cumbersome and thus omitted here. They can easily be obtained from Eq. (11) as it was described above.

It is worth mention, that all eigenvectors in Fig. 3 are universal. They can be found using the group approach as it is described in standard textbooks on molecular vibrations

[25]. These eigenvectors are independent of the concrete form of mutual potential. In contrast, most of normal displacements depend on the specific form of $U(r)$ for all clusters beginning with the pentagon. Thus these clusters are of less concern to experimental measurements.

To obtain the concrete results for the repulsive Yukawa potential one should substitute Eq. (16) into the expressions of this section. We have also considered a generalized Yukawa potential with the long-range attraction, which is described by Eq. (2). The expressions for the frequencies are presented in the Appendix. Evidently, more general potentials can also be easily considered.

V. CONCLUSION

In conclusion let us summarize our results. We have considered a polygonal pattern of dusty grains in the parabolic confining potential wall. The mutual interaction is assumed to be described by a certain potential function $U(r)$. Assuming that the polygonal solution exists, we have found the stability criterion, the eigenmodes and their frequencies for an arbitrary form of mutual interaction. The exact expressions for the frequencies will allow us to restore some information on $U(r)$. In particular, the expressions for the frequencies listed in the Appendix, can help measure the possible long-range attraction of the dusty grains.

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APPENDIX

Here we list the frequencies of the polygon modes. We assume the interaction of dusty grains be described by Eq. (2). The results for the pure Yukawa potential can be obtained by setting $\alpha=0$.

The expressions are written in terms of dimensionless variables. The frequencies are normalized by $\nu=\sqrt{q^2/(mR^3)}$, the normalized frequencies are denoted by tilde. The distances are normalized by the Debye length, we use the dimensionless parameter $\lambda=R/r_d$.

1. Two particles

The equilibrium condition is

$$\tilde{\omega}_0^2=\frac{1}{4}(1+2\lambda)e^{-2\lambda}-\frac{1}{4}\alpha.$$

The frequency of radial oscillations is

$$\tilde{\omega}^2=3\tilde{\omega}_0^2+\lambda^2e^{-2\lambda}.$$

2. Triangle

The equilibrium condition is

$$\tilde{\omega}_0^2=\frac{1}{\sqrt{3}}(1+\sqrt{3}\lambda)e^{-\sqrt{3}\lambda}-\frac{1}{\sqrt{3}}\alpha.$$

The frequency of radial oscillations is

$$\tilde{\omega}^2=3\tilde{\omega}_0^2+\sqrt{3}\lambda^2e^{-\sqrt{3}\lambda}.$$

The frequency of the remaining doubly degenerate mode is

$$\tilde{\omega}^2=\frac{3}{2}\tilde{\omega}_0^2+\frac{\sqrt{3}}{2}\lambda^2e^{-\sqrt{3}\lambda}.$$

3. Square

We introduce notation

$$\tilde{\omega}_a^2=\frac{1}{\sqrt{2}}(1+\sqrt{2}\lambda)e^{-\sqrt{2}\lambda}-\frac{1}{\sqrt{2}}\alpha,$$

$$\tilde{\omega}_b^2=\frac{1}{4}(1+2\lambda)e^{-2\lambda}-\frac{1}{4}\alpha.$$

Then the equilibrium condition becomes

$$\tilde{\omega}_0^2=\tilde{\omega}_a^2+\tilde{\omega}_b^2.$$

The frequency of the radial oscillations is

$$\tilde{\omega}^2=3\tilde{\omega}_0^2+\lambda^2(\sqrt{2}e^{-\sqrt{2}\lambda}+e^{-2\lambda}).$$

The frequency of the doubly degenerate mode is

$$\tilde{\omega}^2=2\tilde{\omega}_a^2+\tilde{\omega}_b^2+\sqrt{2}\lambda^2e^{-\sqrt{2}\lambda}.$$

The frequencies of the remaining singly degenerate modes are

$$\tilde{\omega}^2=3\tilde{\omega}_b^2+\lambda^2e^{-2\lambda},$$

$$\tilde{\omega}^2=3\tilde{\omega}_a^2+\sqrt{2}\lambda^2e^{-\sqrt{2}\lambda}.$$

[1] V.N. Tsytovitch, Usp. Fiz. Nauk **167**, 57 (1997).
[2] J.H. Chu and I. Lin, Phys. Rev. Lett. **72**, 4009 (1994).
[3] H. Thomas, G. Morfill, V. Demmel, J. Goree, B. Feuerbacher, and D. Möhlmann, Phys. Rev. Lett. **73**, 652 (1994).
[4] Y. Hayashi and K. Tachibana, Jpn. J. Appl. Phys., Part 1 **33**, 804 (1994).
[5] A. Melzer, T. Trottenberg, and A. Piel, Phys. Lett. A **191**, 301 (1994).
[6] R.A. Quinn, C. Cui, J. Goree, J.B. Pieper, H. Thomas, and G.E. Morfill, Phys. Rev. E **53**, R2049 (1996).
[7] J.B. Pieper and J. Goree, Phys. Rev. Lett. **77**, 3137 (1996).
[8] J. Perrin, P. Molinas-Mata, and Ph. Belenguer, J. Phys. D **27**, 2499 (1994).
[9] S.V. Vladimirov and M. Nambu, Phys. Rev. E **52**, R2172 (1995).
[10] Lin I, Y. J. Lai, W. T. Juan, and M. H. Chen, in *Frontiers in*

Dusty Plasmas, edited by Y. Nakamura, T. Yokota, and P. K. Shukla (Elsevier, New York, 2000), p. 75.
[11] J. Goree, D. Samsonov, Z. W. Ma, A. Bhattacharjee, H. M. Thomas, U. Konopka, and G. E. Morfill, in *Frontiers in Dusty Plasmas*, edited by Y. Nakamura, T. Yokota, and P. K. Shukla (Elsevier, New York, 2000), p. 91.
[12] Y. Chen, H. Luo, M. Ye, and M. Yu, Phys. Plasmas **6**, 699 (1999).
[13] O. Ishihara, Phys. Plasmas **5**, 357 (1998).
[14] Kelvin, in *Hydrodynamics and General Dynamics*, Mathematical and Physical Papers Vol. 4 (Cambridge University Press, Cambridge, 1910).
[15] J. J. Thomson, *On the Motion of Vortex Rings* (MacMillan, London, 1883).
[16] C.F. Driscoll and K.S. Fine, Phys. Fluids B **2**, 1359 (1990).
[17] L.J. Campbell, Phys. Rev. A **24**, 514 (1981).
[18] D.J. Wineland, Science **226**, 395 (1984).
[19] H. Walther, Adv. At., Mol., Opt. Phys. **31**, 137 (1993).
[20] V.M. Bedanov and F.M. Peeters, Phys. Rev. B **49**, 2667 (1994).
[21] V.A. Schweigert and F.M. Peeters, Phys. Rev. B **51**, 7700 (1995).
[22] R. Rafac, J.P. Schiffer, J.S. Hangst, D.H.E. Dubin, and D.J. Wales, Proc. Natl. Acad. Sci. U.S.A. **88**, 483 (1991).
[23] I.V. Schweigert, V.A. Schweigert, and F.M. Peeters, Phys. Rev. B **54**, 10 827 (1996).
[24] Sh. Amiranashvili, N. Gusein-zade, and A. Ignatov, Phys. Rev. A **59**, 3098 (1999).
[25] J.P. Elliott and P.G. Dawber, *Symmetry in Physics* (Macmillan Press, London, 1979), Vols. I and II.